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Abstract This is a study of the representations of subjective expected utility preferences that admit state-dependent incompleteness, and subjective expected utility preferences displaying non-comparability of acts from distinct sources. The notions familiar events and sources are defined and characterized. The relation greater familiarity on sources and increasing familiarity of a source are also defined and characterized.

Keywords Incomplete preferences · Source familiarity · Event familiarity · State-dependent incomplete preferences

JEL Classification D81

> "It is conceivable—and may even in a way be more realistic—to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable" (Neumann and Morgenstern 1947).

> "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint" (Aumann 1962).

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1 Introduction

In general, the representations of subjective expected utility theory with incomplete preferences have the form of multi-prior expected multi-utility.[1] The set of priors represents the incompleteness of beliefs, and the set of utility functions represents that of tastes. Specific models admit complete preferences on a subset of acts leading to more restrictive representations. For example, in Bewley's (2002) model of Knightian uncertainty, the restriction of the preference relation to constant acts is complete, giving rise to multi-prior expected utility representation. In Galaabaatar and Karni (2013), the axiom of complete beliefs implies that if the preference relation between bets and constant acts is complete, then its representation takes the form of subjective expected multi-utility representation.

This paper explores the underlying structures of incomplete preference relations under uncertainty that are complete over distinct classes of acts. The motivation for this inquiry is the presumption that some classes of acts share features that make them more readily comparable, while acts belonging to distinct classes are not comparable.

To begin with, I examine subsets of acts that agree outside a given event and display event-dependent incompleteness. The interest in this investigation stems from the perception that the conditional preferences may be incomplete due to lack of familiarity with the underlying events. Event unfamiliarity might impact the decision maker's confidence in his own tastes when facing a choice among acts that agree outside the said event. This type of situations is prevalent in medical decision-making. Consider, for example, a person diagnosed as having prostate cancer and must choose between alternative treatments, say surgery and radiation therapy. Suppose that the patient is informed about the likelihood of being cured and the probabilities of other potential outcomes, including incontinence and impotency, associated with each of these treatments. Conceivably, the patient believes the likelihoods of the different outcomes under the different treatments as facts and yet finds it difficult to express clear preferences among the treatments because the potential outcomes include states of health that he has never experienced. In other words, it is quite natural that the patient is not clear about his own preferences conditional on the unfortunate events in which one of the bad outcomes obtains. In this instance, the indecisiveness is due to incompleteness of tastes rather than that of beliefs.

Situations that require choice among acts whose payoffs are contingent on events that are more and less familiar are both important and prevalent. Other than medical decision-making, these situations include decisions about health insurance, long-term care insurance, disability insurance, career choice, and choice of education, to mention but a few examples. In all of these examples, it is reasonable to suppose that the probabilities of the relevant events are given by the relative frequencies of their occurrence in the population, and the incompleteness of the preferences is attribute to incompleteness of tastes.

[1] See Seidenfeld et al. (1995), Nau (2006) and Galaabaatar and Karni (2013).

A different type of acts on which the preference relation may be complete, or incomplete to different degrees, are sets of acts belonging to a "familiar" source.[2] It has been suggested that source preference, or familiarity bias, might explain the observed tendency of investors to forego portfolio diversification in order to invest in what they perceive to be more familiar companies (Heath and Tversky 1991) or more familiar institutional environment, leading to domestic bias in financial and other investments, (Huberman 2001). Rigotti and Shannon (2005) explore the general equilibrium implications, including the indeterminacy of the equilibrium prices and the allocation efficiency, due to incomplete beliefs.[3] More recently, Chew et al. (2012) provide genetic evidence in support of the view that a sense of familiarity or competence or lack of them underlies both ambiguity aversion and familiarity bias. In these instances, it is reasonable to suppose that tastes regarding the payoffs of the investments are complete and that source unfamiliarity underlies the incompleteness of beliefs.

Building upon Galaabaatar and Karni (2013), the main results of this paper are representations of incomplete preferences whose restrictions to familiar events and sources are complete. The study of representations of incomplete preferences due to lack of familiarity of events is the subject matter of the next section. Section 3 explores the representations of incomplete preferences due to lack of familiarity of sources. Concluding remarks appear in Sect. 4. The proofs are collected in Sect. 5.

2 Conditionally complete preferences

2.1 The analytical framework and basic preference structure

Invoking the analytical framework of Anscombe and Aumann (1963), let S be a finite set of *states* and denote X be a set of *outcomes*. Let $\Delta(X)$ denote the set of all simple probability measures on X and, for each $x \in X$, denote by δ_x the element of $\Delta(X)$ that assigns the outcome x the unit probability mass.[4] Let H be the set of all functions from S to $\Delta(X)$. Elements of H are referred to as *acts*. For every $p \in \Delta(X)$, h^p denotes the constant act whose payoff is p (that is, $h^p(s) = p$ for all $s \in S$). For each $\ell, \ell' \in \Delta(X)$ and $\alpha \in [0, 1]$ define $\alpha\ell + (1-\alpha)\ell' \in \Delta(X)$ by $\left(\alpha\ell + (1-\alpha)\ell'\right)(x) = \alpha\ell(x) + (1-\alpha)\ell'(x)$ for all $x \in X$. For all $h, h' \in H$ and $\alpha \in [0, 1]$, define $\alpha h + (1-\alpha)h' \in H$ by $\left(\alpha h + (1-\alpha)h'\right)(s) = \alpha h(s) + (1-\alpha)h'(s)$ for all $s \in S$, where the convex mixture $\alpha h(s) + (1-\alpha)h'(s)$ is defined as above. Under this definition, H is a convex subset of the linear space $\mathbb{R}^{|X|\cdot|S|}$. Subsets of S are referred to as *events*.

Let $\succ$ be a binary relation on H. The set H is said to be $\succ$-bounded if there exist h^M and h^m in H such that $h^M \succ h \succ h^m$ for all $h \in H - \{h^M, h^m\}$.

The following axioms depict the basic structure of the *preference relation* $\succ$ and are maintained throughout. These axioms are well-known and require no elaboration.

[2] See Chew and Sagi (2008) for a model of attitudes toward source uncertainty.

[3] They describe familiar sources as risky events and unfamiliar sources as uncertain events.

[4] A probability measure is simple if it has a finite support.

(A.1) (**Strict partial order)** The preference relation $\succ$ is transitive and irreflexive.
(A.2) (**Archimedean)** For all $f, g, h \in H$, if $f \succ g$ and $g \succ h$ then $\beta f + (1-\beta) h \succ g$ and $g \succ \alpha f + (1-\alpha) h$ for some $\alpha, \beta \in (0, 1)$.
(A.3) (**Independence**) For all $f, g, h \in H$ and $\alpha \in (0, 1]$, $f \succ g$ if and only if $\alpha f + (1-\alpha) h \succ \alpha g + (1-\alpha) h$.

The difference between the preference structure above and that of expected utility theory is that the induced relation $\neg (f \succ g)$ is reflexive, but not necessarily transitive (hence, it is not necessarily a preorder). Moreover, $\neg (f \succ g)$ and $\neg (g \succ f)$, does not imply that f and g are indifferent, rather they may be *incomparable*.

For every $h \in H$, denote by $B(h) := \{f \in H \mid f \succ h\}$ and $W(h) := \{f \in H \mid h \succ f\}$ the (strict) upper and lower contour sets of h, respectively. The relation $\succ$ is said to be convex if the upper contour set is convex. If H is $\succ$-bounded, then for $h \neq h^M, h^m$, $B(h)$ and $W(h)$ have nonempty algebraic interior in the linear space generated by H. A binary relation on $\succ$ on H satisfying (A.1)–(A.3) is convex. Moreover, the lower contour set is also convex.[5]

2.2 Event completeness and conditional dominance

To formalize the idea of conditionally complete preferences, I use the following notations. For every event E and acts h and h' define the act $h_E h'$ by $h_E h'(s) = h(s)$ if $s \in E$ and $h_E h'(s) = h'(s)$, otherwise. Denote by $\succ_E$ the restriction of $\succ$ to the subset of act $\{h_E h' \mid h \in H\}$. Note that (A.1)–(A.3) imply that $\succ_E$ is well-defined (that is, it is independent of h').[6] Define the weak conditional preference relation $\succcurlyeq_E$ on H as follows: For all $f, g \in \{h_E h' \mid h \in H\}$, $f \succcurlyeq_E g$ if $\neg(g \succ_E f)$.

Familiarity of events is a subjective attribute, revealed by choice. Since this term is used throughout the remaining of this work, it is worth elaborating on. The literature on decision-making under uncertainty invokes the term ambiguity to describe events about which a decision makers beliefs may not be depicted by a unique probability. According to this usage, familiarity may be regarded as interpretation of the fact that a decision maker's choice behavior reveals that he regards the events under consideration unambiguous. Similarly, ambiguous events are interpreted as event with regard to which the decision maker experiences lack of familiarity. However, in this work, I allow the source of ambiguity of an event to include the possibility that a decision maker's tastes (e.g., risk attitudes) may not be captured by a unique utility function. Familiarity of an event, in the sense used here, encompasses the lack of ambiguity as well as lack of clarity regarding the valuation of the payoffs contingent on the familiar event. I assume that familiarity of an event E is equivalent to the completeness of the (weak) preference relation over acts that agree on the unfamiliar event, $E^c := S - E$. Formally,

Definition 1 an event, E, is said to be *familiar* if the conditional preference relation $\succ_E$ is negatively transitive.

[5] The proof is by two applications of (A.3).

[6] See Galaabaatar and Karni (2013) for a proof that (A.1)–(A.3) implies that $\succ$ has additive separable representation. The additive separability across states implies that $\succ_E$ is well defined.

If E is a familiar event, then $\succcurlyeq_E$ is complete and transitive.[7]

The next axiom requires that conditional on familiar events, the preference relation satisfies state independence. Formally,

(A.4) (**Conditional state independence**) For every familiar event, E and all $s, s' \in E$, $h \in H$, and $p, q \in \Delta(X)$, $h_{-s}p \succ h_{-s}q$ if and only if $h_{-s'}p \succ h_{-s'}q$.

If $E \subset S$ is familiar and S itself is not then, for some $h', h'' \in H$, the acts $h_E h'$ and $h_E h''$ are noncomparable. The incompleteness of the conditional preference relation $\succ_{E^c}$ may itself be due to incompleteness of beliefs regarding the likelihoods of the different states in the unfamiliar event and/or to the decision maker's lack of confidence in his own tastes in this event. To represent preference relations that are incomplete on some events, I adopt a modified version of the dominance axiom of Galaabaatar and Karni (2013). Let E be a familiar event. For each $f \in H$, let f^s denote the constant act whose payoff is $f(s)$in every state (that is, $f^s\left(s'\right) = f(s)$ for all $s' \in S$). The axiom asserts that if an act, $g_{E^c}h$, is strictly preferred over every act, $f^s_{E^c}h$, for every $s \in E^c$, then $g_{E^c}h$ is strictly preferred over $f_{E^c}h$. To grasp the intuition underlying this assertion, note that for any possible consequence of f in the event E^c, the act $f^s_{E^c}h$ is an element of the lower contour set of $g_{E^c}h$. Convexity of the lower contour sets implies that any convex combination of the consequences of $f_{E^c}h$ is dominated by $g_{E^c}h$. Think of $f_{E^c}h$ as representing a set of such combinations whose elements correspond to the implicit set of conditional subjective probability distributions on E^c that the decision maker might entertain. Since any such combination is dominated by $g_{E^c}h$, so is $f_{E^c}h$. Formally,

(A.5) (**Conditional dominance**) For every familiar event, E, and all $f, g, h \in H$, if $g_{E^c}h \succ f^s_{E^c}h$ for every $s \in E^c$, then $g_{E^c}h \succ f_{E^c}h$.

Theorem 1 below asserts that a preference relation satisfies the axioms (A.1)–(A.5) if and only if there is a nonempty set of utility functions on X and, corresponding to each utility function, a set of probability measures on S such that when facing a choice between two acts, the decision maker prefers the act that yields higher expected utility according to every utility function and every probability measure in the corresponding set. Moreover, if E is a familiar event, then the conditional preference relation, $\succ_E$, has a subjective expected utility representation, and for the corresponding unfamiliar event, E^c, the conditional preference relation, $\succ_{E^c}$, has a multi-prior expected multi-utility representation.

To state the theorem, I use the following notations: Given a familiar event, E, let $\Phi(E) := \{(\pi, v) \mid v \in \mathcal{V}(E), \pi \in \Pi^v\}$, where $\mathcal{V}(E)$ is a nonempty set of real-valued functions on X, and for each $v \in \mathcal{V}(E)$, Π^v is a set of full-support probability measures on S. For every real-valued function, u on X, and probability measure $p \in \Delta(X)$, $u \cdot p := \sum_{x \in X} p(x)u(x)$.

Theorem 1 (Representation: Existence) *Let $\succ$ be a binary relation on H, then the following conditions are equivalent:*

[7] See Kreps (1988) proposition (2.4).

(i) H is $\succ$-bounded and $\succ$ satisfies (A.1)–(A.5).

(ii) For every familiar event, E, there exists a real-valued function $\hat{v}$ on X and a set, $\Phi(E)$, of probability-utility pairs such that, for all $h \in H - \{h^M, h^m\}$ and $(\pi, v) \in \Phi(E)$,

$$\sum_{s\in E}\pi(s)\left(h^M(s)\cdot\hat{v}\right)+\sum_{s\in E^c}\pi(s)\left(h^M(s)\cdot v\right)>\sum_{s\in E}\pi(s)\left(h(s)\cdot\hat{v}\right)$$
$$+\sum_{s\in E^c}\pi(s)\left(h(s)\cdot v\right)>\sum_{s\in E}\pi(s)\left(h^m(s)\cdot\hat{v}\right)+\sum_{s\in E^c}\pi(s)\left(h^m(s)\cdot v\right)$$

and, for all $h, h' \in H$, $h \succ h'$ if and only if

$$\sum_{s\in E}\pi(s)\left(h(s)\cdot\hat{v}\right)+\sum_{s\in E^c}\pi(s)\left(h(s)\cdot v\right)>\sum_{s\in E}\pi(s)\left(h'(s)\cdot\hat{v}\right)$$
$$+\sum_{s\in E^c}\pi(s)\left(h'(s)\cdot v\right), \tag{1}$$

for all $(\pi, v) \in \Phi(E)$.

Moreover, for all $\pi, \pi' \in \cup_{v\in\mathcal{V}}\Pi^v$ and every familiar event, E, the conditional probability measures, $\pi(\cdot \mid E)$ and $\pi'(\cdot \mid E)$, satisfy $\pi(\cdot \mid E) = \pi'(\cdot \mid E)$.

The preference relation depicted above may be thought of as displaying event-dependent incompleteness. This interpretation entails an equivalent formulation of the representation in (1). Let $\mathcal{U}(E) : \{u(\cdot; s) : X \to \mathbb{R} \mid s \in S\}$ be a set of state-dependent utility functions, each of which depends only on the event E (that is, $u(\cdot; s) = u\left(\cdot; s'\right)$ for all $s, s' \in E$, and likewise for the complementary event E^c). Moreover, all the elements of $\mathcal{U}(E)$ agree on E (that is, $u(\cdot; s) = u'(\cdot; s)$ for all $u, u' \in \mathcal{U}(E)$ and $s \in E$). Then,

$$h \succ h' \Leftrightarrow \sum_{s\in S}\pi(s)\left(h(s)\cdot u(\cdot; s)\right)>\sum_{s\in S}\pi(s)\left(h'(s)\cdot u(\cdot; s)\right), \tag{2}$$

for all $(\pi, u) \in \{(\pi, u) \mid u \in \mathcal{U}(E), \pi \in \Pi^u\}$, for each $u \in \mathcal{U}(E)$, Π^u is a set of full-support probability measures on S.

The idea of the proof is as follows: First, use (A.1)–(A.4), the completeness of the weak preference relation on familiar events, and the theorem of Anscombe and Aumann (1963) to obtain a subjective expected utility representation of the preference relation conditional on the familiar event, E. Second, invoke (A.5) and modify the arguments in the proof of Theorem 1 in Galaabaatar and Karni (2013) to obtain a multi-prior expected multi-utility representation of the preference relation conditional on the complementary event, E^c. Third, use the set of probability measures to link the two representations.

To state the uniqueness properties of the utilities and probabilities that figure in the representation in Theorem 1, I introduce the following additional notations. Let $\mathcal{U}$ be a set of real-valued functions on $X \times S$. Fix $x^0 \in X$ and for each $u \in \mathcal{U}$ define a real-valued function, $\hat{u}$, on $X \times S$ by $\hat{u}(x, s) = u(x, s) - u\left(x^0, s\right)$. Let $\widehat{\mathcal{U}}$ be the normalized set of functions corresponding to $\mathcal{U}$ (that is, $\widehat{\mathcal{U}} = \{\hat{u} \mid u \in \mathcal{U}\}$). Let δ_s be the vector in $\mathbb{R}^{|X|\cdot|S|}$ such that $\delta_s(t, x) = 0$ for all $x \in X$ if $t \neq s$ and $\delta_s(t, x) = 1$ for all $x \in X$ if $t = s$. Define $D = \{\theta\delta_s \mid s \in S, \theta \in \mathbb{R}\}$. Denote by $\langle\widehat{\mathcal{U}}\rangle$ the closure of the convex cone in $\mathbb{R}^{|X|\cdot|S|}$ generated by all the functions in $\widehat{\mathcal{U}}$ and D.

For each $(\pi, v) \in \Phi(E)$ that figure in the representation, define a vector $w := (\pi(s) v(x))_{(s,x)\in S\times X}$ in $\mathbb{R}^{|X|\cdot|S|}$. Denote by $\mathcal{W}$ the set of all these vectors. Define $\langle\widehat{\Phi(E)}\rangle = \langle\widehat{\mathcal{W}}\rangle$.

Theorem 2 (Representation: Uniqueness) *If $\Phi'(E) = \{\left(\pi', v'\right) \mid v' \in \mathcal{V}, \pi' \in \Pi^{v'}\}$ and $\hat{v}'$ represent $\succ$ in the sense of (1), then $\langle\widehat{\Phi'(E)}\rangle = \langle\widehat{\Phi(E)}\rangle$, and $\hat{v}'$ is a positive affine transformation of $\hat{v}$.*

The uniqueness result is implied by Lemma 2 in Galaabaatar and Karni (2013). Clearly, the conditional probability measures satisfy $\pi^w(s \mid E) = \hat{\pi}(s) = \pi^{w'}(s \mid E)$, for all $s \in E$ and $w, w' \in \mathcal{W}$. Thus, $\pi(\cdot \mid E) = \pi'(\cdot \mid E) = \hat{\pi}(\cdot) / \Sigma_{s'\in E}\hat{\pi}\left(s'\right)$, for every familiar event, E. The uniqueness of $\hat{v}$ is implied by the uniqueness part of the von Neumann–Morgenstern expected utility theorem.

In general, $\hat{v}$ is not an element of $\mathcal{V}(E)$. However, if $h^p \succ_{E^c} h^q$ implies that $h^p \succ_E h^q$, for all $p, q \in \Delta(X)$, then it is easy to see that $\hat{v} \in \mathcal{V}(E)$.

2.3 Complete beliefs

There are situations in which decision makers rely on experts' assessment of the likelihood of events. For example, in the medical decision problem described in the introduction, the likelihoods of the different outcomes under alternative treatments are provided by the physician. Similarly, accident risks (e.g., airplane crash) are depicted by their empirical distributions. It is reasonable to suppose that in such cases, a decision maker's beliefs coincide with the empirical distributions and are complete. At the same time, the decision maker may feel unable to compare certain acts whose payoffs are contingent on events outside his realm of experience. For instance, a decision maker who enjoyed good health all his life may find it difficult to assess the relative merits of long-term care insurance policies that include payoffs in the events in which he is disabled and needs professional care.

To model situations of involving complete beliefs and incomplete tastes regarding payoffs contingent on unfamiliar events, I invoke the axiom, dubbed complete beliefs, introduced by Galaabaatar and Karni (2013). Denote by $h^p\alpha h^q$ the constant act whose payoff is $\alpha p + (1-\alpha) q$ in every state.

(A.6) (**Complete beliefs**) For all events E and $\alpha \in [0, 1]$, and constant acts h^p and h^q such that $h^p \succ h^q$, either $h^p\alpha h^q \succ h^p E h^q$ or $h^p E h^q \succ h^p\alpha' h^q$, for every $\alpha > \alpha'$.

The next theorem characterizes the representations of preference relations displaying complete beliefs in the presence of unfamiliar events.

Theorem 3 *Let $\succ$ be a binary relation on H, then the following conditions are equivalent:*

(i) H is $\succ$-bounded and $\succ$ satisfies (A.1)–(A.6).
(ii) There is a unique probability measure π on S, a real-valued function, $\hat{v}$ on X and for every familiar event, E a nonempty set, $\mathcal{V}(E)$, of real-valued functions on X, such that, for all $h \in H - \{h^M, h^m\}$ and $v \in \mathcal{V}(E)$,

$$\sum_{s\in E}\pi(s)\left(h^M(s)\cdot\hat{v}\right)+\sum_{s\in E^c}\pi(s)\left(h^M(s)\cdot v\right)>\sum_{s\in E}\pi(s)\left(h(s)\cdot\hat{v}\right)$$
$$+\sum_{s\in E^c}\pi(s)\left(h(s)\cdot v\right)>\sum_{s\in E}\pi(s)\left(h^m(s)\cdot\hat{v}\right)+\sum_{s\in E^c}\pi(s)\left(h^m(s)\cdot v\right), \tag{3}$$

and, for all $f, g \in H$, $f \succ g$ if and only if

$$\sum_{s\in E}\pi(s)\left(f(s)\cdot\hat{v}\right)+\sum_{s\in E^c}\pi(s)\left(f(s)\cdot v\right)>\sum_{s\in E}\pi(s)\left(g(s)\cdot\hat{v}(s)\right)$$
$$+\sum_{s\in E^c}\pi(s)\left(g(s)\cdot v\right), \tag{4}$$

for all $v \in \mathcal{V}(E)$.

Moreover, $\hat{v}$ is unique up to positive affine transformation, and if $\mathcal{U}(E)$ is another set of functions that represent $\succ$ in the sense of (4) then $\langle\widehat{\mathcal{U}}(E)\rangle = \langle\widehat{\mathcal{V}}(E)\rangle$.

The proof of Theorem 3 follows from Theorem 1 above and Theorem 4 of Galaabaatar and Karni (2013), and is omitted.

3 Source familiarity

3.1 Familiar sources

There is a substantial body of literature dedicated to the proposition that decision makers display source preference. In other words, decision makers prefer acts whose payoffs are contingent on events that belong to more familiar sources over acts whose payoffs are contingent on events that belonging to less familiar sources. To formalize the idea of source familiarity, define *source* to be a partition of S.[8] For every partition $\mathcal{T} = \{E_1, \ldots, E_n\}$ of S, let $H(\mathcal{T})$ be the subset of acts that are $\mathcal{T}$-measurable.[9] For every partition $\mathcal{T}$, $H(\mathcal{T})$ with the usual mixture operation is a convex subset of a linear space.[10]

Definition 2 A *familiar source* is a partition, $\mathcal{T}$, of the state space such that $\succ$ on $H(\mathcal{T})$ is negatively transitive. A partition that is not a familiar source is an *unfamiliar source.*

[8] Equivalently, a source may be regarded as the algebra generated by a partition.

[9] That is, $H(\mathcal{T}) = \{h \in H \mid h = \left(p^1_{E_1} p^2_{E_2}, \ldots, p^n_{E^n}\right), p^i \in \Delta(X)$, for all $i = 1, \ldots, n, n < \infty\})$.

[10] The mixture operation is defined as follows: for every $h, f \in H(\mathcal{T})$, and $\alpha \in [0, 1]$, $\alpha h + (1-\alpha) f \in H(\mathcal{T})$ is defined by $(\alpha h + (1-\alpha) f)(s) = \alpha h(s) + (1-\alpha) f(s)$, for all $s \in S$.

Note that, like familiar events, what constitutes a familiar source is subjective perception. It is revealed by the completeness of the weak preference relation of the subset of acts that are measurable with respect to the source. Let $\mathcal{T}^0 := \{S, \varnothing\}$ denote the trivial partition, then $H\left(\mathcal{T}^0\right)$ is the subset of all constant acts.

3.2 Axioms and representation

Consider the axiom of (unconditional) dominance, of Galaabaatar and Karni (2013).

Dominance For all $f, g \in H$, if $g \succ f^s$ for every $s \in S$, then $g \succ f$.

Note that dominance implies that the preference relation satisfies (unconditional) state independence. Then, adding dominance to the (A.1)–(A.3) implies the following multi-prior expected utility theorem.

Theorem 4 *Let $\succ$ be preference relation on H and suppose that $\mathcal{T}^0$ is a familiar source, then the following conditions are equivalent:*

(i) H is $\succ$-bounded and $\succ$ satisfies axioms (A.1)–(A.3) and dominance.
(ii) There is a real-valued function u on X, unique up to positive affine transformation, $\bar{x}, \underline{x} \in X$ such that $h^M = h^{\delta_{\bar{x}}}$, $h^m = h^{\delta_{\underline{x}}}$, and there is a unique, nonempty, convex set, $\mathcal{M}$, of probability measures on S such that, for each source, $\mathcal{T}$, and all $h \in H(\mathcal{T}) - \{h^M, h^m\}$,

$$u(\bar{x}) > \sum_{E \in \mathcal{T}} \pi(E)\,(h(E) \cdot u) > u\left(\underline{x}\right), \forall \pi \in \mathcal{M}. \tag{5}$$

For all h, $h' \in H$,

$$h \succ h' \Leftrightarrow \sum_{s \in S} \pi(s)\,(h(s) \cdot u) > \sum_{s \in S} \pi(s)\left(h'(s) \cdot u\right), \forall \pi \in \mathcal{M}. \tag{6}$$

Moreover, for every familiar source, $\mathcal{T}$, and all $E \in \mathcal{T}$, $\pi(E) = \pi'(E)$ for all $\pi, \pi' \in \mathcal{M}$.

For any unfamiliar source, $\mathcal{T}$ and all $h \in H(\mathcal{T})$, h is represented by $\sum_{E \in \mathcal{T}} \pi(E)\,(h(E) \cdot u)$, $\pi \in \mathcal{M}$. For any familiar source, $\mathcal{T}$ there is a unique source-dependent probability measure $\pi_{\mathcal{T}}$ on S such that for all $h \in H(\mathcal{T})$, h is represented by $\sum_{E \in \mathcal{T}} \pi_{\mathcal{T}}(E)\,(h(E) \cdot u)$. For every element, E, of a partition that defines a familiar source, the subjective probabilities of the states are not unique. However, the sum of the probabilities of all the states in E is unique.

3.3 Comparative source familiarity

Distinct unfamiliar sources are not always unfamiliar to the same degree. To formalize this idea, I define a relation "more familiar than" on the set of sources in terms of the preference relations and characterize it in terms of the set of probabilities.

Following Galaabaatar and Karni (2013), define a binary relation $\succcurlyeq_{GK}$ on H by: For all $f, g \in H$, $f \succcurlyeq_{GK} g$ if $h \succ f$ implies $h \succ g$, for all $h \in H$. If the preference relation is complete, this weak preference relation coincides with the usual definition of weak preferences, namely, the negation of strict preferences. However, if the preference relation is incomplete, the two definitions of weak preferences are distinct. In particular, when the preference relation is incomplete, the traditional definition the symmetric part of the weak preference relation confounds indifference and noncomparability. The symmetric part of the preference relation $\succcurlyeq_{GK}$ is an indifference relation, distinct from noncomparability.[11] It is possible to show that all the representation results in this paper apply to $\succcurlyeq_{GK}$, with weak inequalities replacing the strict inequalities.[12]

For every event E, a *bet on* E is the an act, $x_E y$, as follows: $(x_E y)(s) = \delta_x$ if $s \in E$ and $(x_E y)(s) = \delta_y$, otherwise, where $x, y \in X$, satisfy $h^{\delta_x} \succ h^{\delta_y}$. Denote by $\sim_{GK}$ the symmetric part of $\succcurlyeq_{GK}$. An event is said to be *null* if $x_E y \sim_{GK} x'_E y$, for all $x, x' \in X$. An event E is *nonnull* if it is not null. For every $E \subset S$, let $\bar{\pi}(E) := \sup\{\pi(E) \mid \pi \in \mathcal{M}\}$ and $\underline{\pi}(E) := \inf\{\pi(E) \mid \pi \in \mathcal{M}\}$. Clearly, if E is nonnull then $\bar{\pi}(E) > 0$.

To define the relation "more familiar than," I use the following notations and definitions: Let $\rhd$ be a binary relation on H defined as follows: For all $f, g \in H$, $f \rhd g$ if $f \succcurlyeq_{GK} g$ and $\neg(f \succ g)$. In terms of the representation $f \rhd g$ means that the subjective expected utility of f exceeds that of g according to every utility–probability pair that figure in the representation except one. Put differently, f is an element on the boundary of the upper contour set of g.

Fix a constant act h^p and let bets $x_E y$ and $\hat{x}Ey$ such that $x_E y \rhd h^p \rhd \hat{x}_E y$. Then, by definition of $\rhd$, the bet $x_E y$ is in the boundary of the upper contour set of h^p, and the bet $\hat{x}_E y$ is in the boundary of the lower contour set of h^p. Since the payoff, y, of both bets in the complementary event, E^c, is the same, the pair of outcomes $(x, \hat{x})$ spans of the gulf between the upper and lower contour sets of h^p. The presumption is that the less familiar is the source, the bigger is the gulf that needs to be spanned, for every bet on event E' belonging to that source.[13] To make sure that the measurements of these gulfs is based on the same scale, the "upper bet," xEy, is fixed. Formally,

Definition 3 A source $\mathcal{T}$ is *more familiar than* a source $\mathcal{T}'$ if for all nonnull $E \in \mathcal{T}$ and $E' \in \mathcal{T}'$ and all $\delta_y, \delta_x, p, p' \in \Delta(X)$, $xEy \rhd h^p \rhd \hat{x}Ey$, and $xE'y \rhd h^{p'} \rhd \hat{x}'E'y$ imply that $\delta_{\hat{x}'} \prec \delta_{\hat{x}}$.[14]

The following theorem characterizes the relation "more familiar than" in terms of the probability measures that figure in the representation. This characterization entails a *monotone likelihood ratio property*, namely, one source is more familiar than another if the likelihood ratio of the lowest, and highest probabilities of each event belonging

[11] Karni (2011) discusses the implications of the weak preference relations, $\succcurlyeq_{GK}$ and $\succcurlyeq$ for the continuity that may be imposed if the strict preference relation is incomplete.

[12] See Galaabaatar and Karni (2013).

[13] Clearly, if the preference relation is complete then $x = \hat{x}$.

[14] Because E and E' are elements of distinct sources. If xEy is in the boundary of the upper contour sets of the constant acts h^p and $h^{p'}$, p and p' are not necessarily the same.

to the more familiar source is larger than the corresponding likelihood ratio of every event belonging to the less familiar source.

Theorem 5 *Let $\succ$ be an Archimedean strict partial order on H satisfying independence and dominance. Suppose that H is $\succ$-bounded and that $\mathcal{T}^0$ is a familiar source. Then source $\mathcal{T}$ is more familiar than another source, $\mathcal{T}'$, if and only if, for all nonnull $E \in \mathcal{T}$ and $E' \in \mathcal{T}'$,*

$$\frac{\underline{\pi}(E)}{\bar{\pi}(E)} > \frac{\underline{\pi}(E')}{\bar{\pi}(E')}.$$

3.4 Increasing source familiarity

Presumably, familiarity grows with experience. As sources become more familiar the preference relation become less incomplete. The following definition captures this idea.

Definition 4 A source $\mathcal{T}$ is *more familiar according to the preference relation* $\succ$ *than according to the preference relation* $\succ'$ if $f \succ' g$ implies $f \succ g$, for all $f, g \in H(\mathcal{T})$.

The next theorem characterizes the notion that a source is more familiar according to one preference relation than according to another. Let $\mathcal{M}^{\succ}$ denote the set of probability measures on S that figure in the representation of $\succ$ in theorem 4.

Theorem 6 *Let $\succ$ and $\succ'$ be Archimedean strict partial orders on H satisfying independence and dominance. Suppose that H is $\succ$ and $\succ'$ bounded, and that $\mathcal{T}^0$ is a familiar source. Then $\mathcal{T}$ is more familiar under $\succ$ than under $\succ'$ if and only if $\mathcal{M}^{\succ} \subseteq \mathcal{M}^{\succ'}$.*

Building on Gilboa and Schmeidler (1989) and departing from their uncertainty aversion axiom, Ghirardato et al. (2004) advance a model of choice in which they separate ambiguity from ambiguity attitude. One of the primitives of their model is a complete preference relations on Anscombe and Aumann (1963) acts. From this, they derive an incomplete preference relations representing "unambiguous" preferences. According to their approach, an act f is (weakly) unambiguously preferred over another act g if and only if the subjective expected utility of the former is at least as great as that of the latter, for every probability measure belonging to a convex and closed set of priors. A derived preference relation $\succcurlyeq_1^*$ is said to reveal more ambiguity than another derived preference relation $\succcurlyeq_2^*$, if $f \succcurlyeq_1^* g$ implies $f \succcurlyeq_1^* g$ for all acts f, g. The resemblance of this to Definition 4 above is apparent. Proposition 6 of Ghirardato et al. (2004) says that $\succcurlyeq_1^*$ reveals more ambiguity $\succcurlyeq_2^*$ if and only if $\mathcal{C}_2 \subseteq \mathcal{C}_1$, where $\mathcal{C}_i$ is the set of priors corresponding to $\succcurlyeq_i^*$, $i = 1, 2$. Clearly, except of context and interpretation, Theorem 6 and Proposition 6 of Ghirardato et al. (2004) are essentially the same.

4 Proofs

4.1 Proof of theorem 1 (outline)

The proof invokes results and arguments from the proof of Theorem 1 in Galaabaatar and Karni (2013), properly modified to accommodate the conditional preferences.

$(i) \Rightarrow (ii)$. For every familiar event, E, the sets of acts that agree on the complementary event (that is, the sets $\{\{f_E h \mid f \in H\} \mid h \in H\}$) are convex.[15] For familiar events, E, the conditional preference relation, $\succcurlyeq_E$, is a weak order (i.e., transitive and complete binary relation) satisfying the weak version of (A.3). Moreover, since $\succ_E \neq \varnothing$, by (A.4) and the theorem of Anscombe and Aumann (1963), there is a real-valued function, $\hat{v}$ on X, and a unique (conditional) probability measure, $\hat{\pi}$ on E, such that, for all $h, g \in H$, $g \succcurlyeq_E f$ if and only if $\sum_{s \in E} \hat{\pi}(s) \sum_{x \in X} \hat{v}(x)\, g(x; s) \geq \sum_{s \in E} \hat{\pi}(s) \sum_{x \in X} \hat{v}(x)\, f(x; s)$. Moreover, $\hat{v}$ is unique up to positive affine transformation, and $\hat{\pi}$ is unique.

Define an auxiliary binary relation $\succcurlyeq$ on H as follows: For all $f, g \in H$, $f \succcurlyeq g$ if $h \succ f$ implies $h \succ g$ for all $h \in H$.[16] Let $B := \{\lambda\left(h' - h\right) \mid h' \succcurlyeq h, h', h \in H, \lambda \geq 0\}$. Each $f \in H$ is a point in $\mathbb{R}^{|X| \cdot |S|}$. However, because the weights on consequences in each state add up to 1, f can also be seen as a point in $\mathbb{R}^{(|X|-1) \cdot |S|}$. For any act $f \in H$ the corresponding act in $\mathbb{R}^{(|X|-1) \cdot |S|}$ is denoted by $\phi(f)$. Thus, $\phi : \mathbb{R}^{|X| \cdot |S|} \to \mathbb{R}^{(|X|-1) \cdot |S|}$ is a one-to-one linear mapping. Define $\phi(B) := \{\lambda \phi(f - h) \mid f \succcurlyeq h, f, h \in H, \lambda > 0\}$. Then $\phi(B)$ is a closed convex cone with nonempty interior in $R^{(|X|-1) \cdot |S|}$. By theorem V.9.8 in Dunford and Schwartz (1957), there is a dense set, T, in its boundary such that each point of T has a unique tangent. Let $\mathcal{W}^o$ be the collection of all the supporting hyperplanes corresponding to this dense set. Without loss of generality, we assume that each function in $\mathcal{W}^o$ has unit normal vector. Then $\mathcal{W}^o$ represents $\succcurlyeq$. Henceforth, let $(w \circ \phi)(f)$ be denoted by $w(f)$, $w \in \mathcal{W}^o$. With this convention, for all $f, g \in H$,

$$f \succcurlyeq_{E^c} g \Leftrightarrow \sum_{s \in E^c} w(f(s), s) \geq \sum_{s \in E^c} w(g(s), s), \forall w \in \mathcal{W}^o.$$

For every $\alpha \in \Delta(S)$, let f^α be the constant act such that $f^\alpha(s) = \Sigma_{s' \in S} \alpha_{s'} f\left(s'\right)$ for all $s \in S$. By an argument analogous to Galaabaatar and Karni (2013), it can be shown that (A.5) is equivalent to the following condition. For all $f, g, h \in H$, $g_{E^c} h \succ (f^\alpha)_{E^c} h$ for every $\alpha \in \Delta(E^c)$ implies that $g_{E^c} h \succ f_{E^c} h$.

The proof that the component functions, $\{w(\cdot, s)\}_{s \in E^c}$, of each function, $w \in \mathcal{W}^o$, are positive linear transformations of one another (that is, if $\hat{w} \in \mathcal{W}^o$, then for all nonnull $s, t \in E^c$, $\hat{w}(\cdot, s)$ and $\hat{w}(\cdot, t)$ are positive linear transformations of one another) is analogous to that of Lemma 6 in Galaabaatar and Karni (2013).

[15] The convex combination $\alpha f_E h + (1 - \alpha) g_E h$ is define, as usual, by $(\alpha f_E h + (1 - \alpha) g_E h)(s)$ is $\alpha f(s) + (1 - \alpha) g(s)$ if $s \in E$ and $h(s)$ if $s \in E^c$.

[16] This relation was first introduced in Galaabaatar and Karni (2013) and was further studied in Karni (2011).

The representation is implied by the following arguments: For each $w \in \mathcal{W}^o$, define $v^w(\cdot) = w(\cdot, t)$ for some $t \in E^c$, and for all $s \in E^c$, let $w(\cdot, s) = b_s^w v^w(\cdot) + a_s^w$, $b_s^w > 0$. Define $\pi^w(s) = \hat{\pi}(s) / (1 + \Sigma_{s' \in E^c} b_{s'}^w)$ for all $s \in E$, and $\pi^w(s) = b_s^w / (1 + \Sigma_{s' \in E^c} b_{s'}^w)$, for all $s \in E^c$. Let $\mathcal{V}(E)$ be the collection of distinct v^w and, for each $v \in \mathcal{V}(E)$, define $\Pi^v = \{\pi^w \mid \forall w \text{ such that } v^w = v\}$.[17]

$(ii) \Rightarrow (i)$. The $\succ$-boundedness. of H and axioms (A.1)–(A.3) are implied by Lemma 2 in Galaabaatar and Karni (2013) and (A.4) and (A.5) are immediate implications of the representation.

Let $\pi^w, \pi^{w'} \in \cup_{v \in \mathcal{V}} \Pi^v$, then, $\pi^w(s \mid E) = \hat{\pi}(s) = \pi^{w'}(s \mid E)$, $\forall s \in S$. Thus, $\pi^w(\cdot \mid E) = \pi^{w'}(\cdot \mid E) = \hat{\pi}(\cdot) / \Sigma_{s' \in E} \hat{\pi}(s')$, for every familiar event E. □

4.2 Proof of theorem 4

(Sufficiency) Since $\mathcal{T}^0$ is a familiar source, the restriction of $\succcurlyeq$ to the subset of constant acts is complete. Hence, by Galaabaatar and Karni (2013), Theorem 3, there is a real-valued function, u on X, unique up to positive affine transformation and a convex nonempty set, $\mathcal{M}$, of probability measures on S such that, for all $h \in H$,

$$\sum_{s \in S} \pi(s) \left(h^M(s) \cdot u\right) > \sum_{s \in S} \pi(s) \left(h(s) \cdot u\right) > \sum_{s \in S} \pi(s) \left(h^m(s) \cdot u\right), \tag{7}$$

for all $\pi \in \mathcal{M}$. Let $\bar{x} \in \arg\max_X u(x)$, $\underline{x} \in \arg\min_X u(x)$, $h^M = \delta_{\bar{x}}$, and $h^m = \delta_{\underline{x}}$. Then $\sum_{s \in S} \pi(s) \left(h^M(s) \cdot u\right) = u(\bar{x})$ and $\sum_{s \in S} \pi(s) \left(h^m(s) \cdot u\right) = u(\underline{x})$. Hence, (5) is implied by (7) and (6) is an implication of Galaabaatar and Karni (2013).

Let $\mathcal{T}$ be a familiar source then $H(\mathcal{T})$ is a convex subset of a linear space. The restriction of $\succcurlyeq$ to $H(\mathcal{T})$ is a weak order satisfying (A.2), the weak version of (A.3) and, by Lemma 2 in Galaabaatar and Karni (2013), it satisfies state independence. Hence, by Anscombe and Aumann (1963), there exists unique $\pi_{\mathcal{T}}$ such that $\mathcal{T}$, for all $f, g \in H(\mathcal{T})$, $f \succ g$ if and only if $\sum_{E \in \mathcal{T}} \pi_{\mathcal{T}}(E) \left(f(E) \cdot u\right) > \sum_{E \in \mathcal{T}} \pi_{\mathcal{T}}(E) \left(g(E) \cdot u\right)$. But, by (6), $f \succ g$ if and only if $\sum_{E \in \mathcal{T}} \pi(E) \left(f(E) \cdot u\right) > \sum_{E \in \mathcal{T}} \pi(E) \left(g(E) \cdot u\right)$, for all $\pi \in \mathcal{M}$. Hence, it must be the case that, for all $E \in \mathcal{T}$, $\pi(E) = \pi_{\mathcal{T}}(E)$ for all $\pi \in \mathcal{M}$.

(Necessity) The necessity is implied by Theorem 3 in Galaabaatar and Karni (2013). □

4.3 Proof of theorem 5

Let $\succ$ be preference relation on H satisfying the conditions in the hypothesis of the theorem. Suppose that H is $\succ$-bounded and that $\mathcal{T}^0$ is a familiar source.

[17] If there are kinks in B, so that there are more than one supporting hyperplane then by the same argument as in Galaabaatar and Karni (2013), there is at least one w that can be expressed as a limit point of sequence $\{w_n\}$ from $\mathcal{W}^o$. Then, the component functions of w are positive linear transformation of one another. Add all those w's to $\mathcal{W}^o$, then the new set of functions will represent $\succ$.

Suppose that $\mathcal{T}$ is a more familiar source than $\mathcal{T}'$. Let $E \in \mathcal{T}$ and $E' \in \mathcal{T}'$ be non-null. Then $\bar{\pi}(E) > 0$ and $\bar{\pi}\left(E'\right) > 0$. Fix a bet on E, $x_E y$. Without loss of generality, normalize the utility function so that $u(y) = 0$. Let $U(p) = \Sigma_{x \in X} u(x) p(x)$. Then, by theorem 4, $x_E y \rhd h^p \rhd \hat{x}_E y$, if and only if

$$\underline{\pi}(E) u(x) = U(p) = \bar{\pi}(E) u\left(\hat{x}\right), \tag{8}$$

and $x_{E'} y \rhd h^{p'} \rhd \hat{x}'_{E'} y$ if and only if

$$\underline{\pi}\left(E'\right) u(x) = U\left(p'\right) = \bar{\pi}\left(E'\right) u\left(\hat{x}'\right). \tag{9}$$

But (8) and (9) imply

$$\frac{\underline{\pi}(E)}{\bar{\pi}(E)} = \frac{u\left(\hat{x}\right)}{u(x)} \text{ and } \frac{\underline{\pi}\left(E'\right)}{\bar{\pi}\left(E'\right)} = \frac{u\left(\hat{x}'\right)}{u(x)}. \tag{10}$$

Now, $\delta_{\hat{x}'} \prec \delta_{\hat{x}}$if and only if $u\left(\hat{x}'\right) < u\left(\hat{x}\right)$. Thus (10) holds if and only if

$$\frac{\underline{\pi}(E)}{\bar{\pi}(E)} > \frac{\underline{\pi}\left(E'\right)}{\bar{\pi}\left(E'\right)}. \tag{11}$$

$\square$

4.4 Proof of theorem 6

Let $\succ$ and $\succ'$ be preference relations on H satisfying the conditions in the hypothesis of the theorem. Suppose that H is $\succ$ and $\succ'$ bounded and that $\mathcal{T}^0$ is a familiar source.

(Necessity) Fix a source $\mathcal{T}$ and let $\succ$ and $\succ'$ be preference relations on H such that $\mathcal{T}$ is more familiar according to $\succ$ than according to $\succ'$. Take any $f, g \in H(\mathcal{T})$ such that $f \succ' g$. By Theorem 4, there exists an real-valued affine function u' on $\Delta(X)$ and a convex set of probability measures, $\mathcal{M}^{\succ'}$ such that

$$\sum_{E \in \mathcal{T}} \pi(E)\left(f(E) \cdot u'\right) > \sum_{E \in \mathcal{T}} \pi(E)\left(g(E) \cdot u'\right), \forall \pi \in \mathcal{M}^{\succ'}. \tag{12}$$

By Definition 4, (12) and Theorem 4, there exists a real-valued affine function u on $\Delta(X)$and a convex set of probability measures, $\mathcal{M}^{\succ}$ such that (12) implies

$$\sum_{E \in \mathcal{T}} \pi(E)(f(E) \cdot u) > \sum_{E \in \mathcal{T}} \pi(E)(g(E) \cdot u), \forall \pi \in \mathcal{M}^{\succ}. \tag{13}$$

But $H\left(\mathcal{T}^0\right) \subseteq H(\mathcal{T})$. Hence, for all $p, q \in \Delta(X)$ and $h^p, h^q \in H\left(\mathcal{T}^0\right)$, $h^p \succ' h^q$ if and only if $p \cdot u' > q \cdot u'$. and $h^p \succ h^q$ if and only if $p \cdot u > q \cdot u$. By Definition 4, for all $p, q \in \Delta(X)$, $h^p \succ' h^q$ implies $h^p \succ h^q$. Hence, u' is a positive linear transformation of u, and can be normalized to be equal to u.

Let $E \in \mathcal{T}$, and $R^{\succ}_{\mathcal{T}}(E) = \{r \in \mathbb{R} \mid r = \pi(E)$ for some $\pi \in \mathcal{M}^{\succ}\}$. Define $\bar{r}' = \sup R^{\succ'}_{\mathcal{T}}(E)$, $\underline{r}' = \inf R^{\succ'}_{\mathcal{T}}(E)$, $\bar{r} = \sup R^{\succ}_{\mathcal{T}}(E)$ and $\underline{r} = \inf R^{\succ}_{\mathcal{T}}(E)$. Fix $p, q \in \Delta(X)$ such that $h^p \succ h^q$. Define a bet on E to be the act, $p_E q$ such that $p_E q(s) = p$ if $s \in E$ and $p_E q(s) = q$, otherwise.

Consider the bets $p_E q$, $p'_E q' \in H(\mathcal{T})$, where $p, p', q, q' \in \Delta(X)$ satisfy $h^p \succ h^{p'} \succ h^{q'} \succ h^q$. Suppose that $p_E q \succ' p'_E q'$ then, by (12)

$$r(p \cdot u) + (1-r)u(q \cdot u) > ru\left(p' \cdot u\right) + (1-r)u\left(q' \cdot u\right), \forall r \in R^{\succ'}_{\mathcal{T}}(E). \quad (14)$$

Suppose further that p, p', q and q' are chosen so as to satisfy

$$\underline{r}'(p \cdot u) + \left(1-\underline{r}'\right)u(q \cdot u) = \underline{r}'u\left(p' \cdot u\right) + \left(1-\underline{r}'\right)u\left(q' \cdot u\right). \quad (15)$$

Then, by Definition 4, (14) implies

$$r(p \cdot u) + (1-r)u(q \cdot u) > ru\left(p' \cdot u\right) + (1-r)u\left(q' \cdot u\right), \forall r \in R^{\succ}_{\mathcal{T}}(E), \quad (16)$$

Moreover, (15) and (16) imply that $\underline{r}' \le \underline{r}$.

For a different choice of bets, by the same argument, it can be shown that $\bar{r}' \ge \bar{r}$. Hence, for all $E \in \mathcal{T}$, $\{\pi(E) \mid \pi \in \mathcal{M}^{\succ}\} \subseteq \{\pi(E) \mid \pi \in \mathcal{M}^{\succ'}\}$. Hence, $\mathcal{M}^{\succ} \subseteq \mathcal{M}^{\succ'}$.

(Sufficiency) Suppose that $\mathcal{M}^{\succ} \subseteq \mathcal{M}^{\succ'}$. For all $f, g \in H(\mathcal{T})$, $f \succ' g$ implies

$$\sum_{E \in \mathcal{T}} \pi(E)(f(E) \cdot u) > \sum_{E \in \mathcal{T}} \pi(E)(g(E) \cdot u), \forall \pi \in \mathcal{M}^{\succ'}. \quad (17)$$

Since $\mathcal{M}^{\succ} \subseteq \mathcal{M}^{\succ'}$, (17) implies

$$\sum_{E \in \mathcal{T}} \pi(E)(f(E) \cdot u) > \sum_{E \in \mathcal{T}} \pi(E)(g(E) \cdot u), \forall \pi \in \mathcal{M}^{\succ}. \quad (18)$$

By Theorem 4, (18) implies $f \succ g$. Thus, by Definition 4, $\mathcal{T}$ is more familiar under $\succ$ than under $\succ'$. □

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